is required. In this case the pair $u_{1,0} = 0$, $v_{1,0} = v_3$ is optimal only in some region which can be constructed by continuing the trajectories w_3 into the past (coinciding in this case with trajectories w_2) up to intersection with the surface $r_1(w) = 0$.

Figure 1 shows a typical trajectory w_3. At the start the position is moved along an ellipse with center at point a and the control v_2 has a constant direction up to the switching point p. After the switching at point p motion takes place along an ellipse with center at point b. The lengths of the segments $(a, 0)$, $(0, b)$ equal unity. At the point p_1 $(\mu - \pi/2 - y = 0)$, lying on set M_3, the first player turns off the velocity by impulse and the position is moved to "hard" contact during a time $\pi/2$.

We fix a certain small number $\varepsilon_1 > 0$ and among the trajectories w_2 we isolate a family $w_{2,\varepsilon,1}$ by the following test. Along any trajectory $w_{2,\varepsilon,1}$ of the family indicated, from the estimate $p_1(\dot{w}_{2,\varepsilon,1}) < 0$ follows the estimate $r_1(w_{2,\varepsilon,1}) \leqslant -\varepsilon_1$, while from the estimate $r_1(w_{2,\varepsilon,1}) > -\varepsilon_1$ follows the equality $p_1(w_{2,\varepsilon,1}) = 0$. Suppose that the trajectories $w_{2,\varepsilon,1}$ occupy a region $W_{\varepsilon,1}$. We state the final result.

Theorem 6.2. The controls $u_1^\circ = u_3 = 0$ and $v_1^\circ = v_2$ realize, in the region $W_{\varepsilon,1} \cap W_2^\circ(\max)$ the time $T_1 = t_\zeta + \pi/2$ and the second player cannot increase this time. This time cannot be lessened by the first player by any pair u, v_2 preserving the inclusion $w \in W_{\varepsilon,1} \cap W_2^\circ(\max)$. If the inclusion indicated is not violated until M_2 is hit, then the motion passes into region C_1 through the boundary $T_1 = \pi/2$ $(t_\zeta = 0)$.

REFERENCES

1. Isaacs, R. P., Differential Games. New York, John Wiley and Sons, Inc., 1965.
2. Krasovskii, N. N., Game Problems on the Contact of Motions. Moscow, "Nauka", 1970.
3. Pozharitskii, G. K., Game problem of the hard contact of two points with an impulse thrust in a linear central field. PMM Vol. 36, №6, 1972.
4. Pozharitskii, G. K., Impulse tracking of a point with bounded thrust. PMM Vol. 37, №2, 1973.

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UDC 62 - 50

ON OPTIMUM SELECTION OF NOISE INTERVALS IN DIFFERENTIAL GAMES OF ENCOUNTER

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We examine differential games of encounter in which the minimizing player observes the game's position on a subset Q of the motion interval $[t^\circ, T]$. The subset Q is formed by the second player during the motion, i.e. he switches on a noise eliminating observation. We pose the problem of optimal noise distribution and solve four examples. A general setting of similar problems was given in [1]. Related problems were examined, for example, in [2, 3].

1. Statement of the problem. Let a differential game of players X and Y be specified on a fixed time interval $[t^\circ, T]$ by the differential equations, constraints, and initial conditions

$$X : \dot{x} = f(x, t, u), \quad u(t) \in U, \quad x(t^\circ) = x^\circ \tag{1.1}$$
$$Y : \dot{y} = g(y, t, v), \quad v(t) \in V, \quad y(t^\circ) = y^\circ$$

The dimensions of the phase vectors x, y and of the control vectors u, v are arbitrary; U, V are sets in the space of vectors u, v; f, g are specified functions.

The conditions for information available to player X are as follows. He knows relations (1.1) and on the whole motion interval he observes the exact value of his own phase vector. Player X observes the opponent's phase vector on a set $Q \subset [t^\circ, T]$, consisting of a fixed number N of closed intervals (the observation intervals) $[a_i, b_i]$, $i = 1, \ldots, N$. The instants a_i, b_i are subject to the constraints

$$t^\circ = a_1 \leqslant b_1 \leqslant \ldots \leqslant a_N \leqslant b_N = T \tag{1.2}$$
$$\vartheta_i = a_{i+1} - b_i, \quad i = 1, \ldots, N-1; \quad \sum_{i=1}^{N-1} \vartheta_i \leqslant \vartheta \leqslant T - t^\circ$$

Consequently, the set $P = [t^\circ, T] \setminus Q$ consists of $N - 1$ open intervals (the noise intervals) (b_i, a_{i+1}), $i = 1, \ldots, N - 1$. We consider that player Y forms a set P during the motion and communicates to player X the scalar signal $q(t)$, $t \in [t^\circ, T]$ at each instant (the last instant of exact observation, see [1])

$$q(t) = t, \quad t \in [a_i, b_i], \quad i = 1, \ldots, N \tag{1.3}$$
$$q(t) = b_i, \quad t \in (b_i, a_{i+1}), \quad i = 1, \ldots, N-1$$

The sets P, subject to constraints (1.2), are said to be admissible. It is obvious that admissible sets P and the time functions (signals) $q(t)$, $t \in [t^\circ, T]$, of form (1.3) are in one-to-one correspondence. Therefore, in the subsequent reasoning the set P is sometimes replaced by a signal $q(t)$ specified on the whole interval $[t^\circ, T]$. We note that the value of signal (1.3) at some fixed instant yields only local information on the structure of set P.

Player X forms his own control at instant t by having available the collection of quantities $z = \{x(t), y(q(t)), t, q(t)\}$, $t \in [t^\circ, T]$, i.e. employs strategies in the form of the functions $u = u(z)$. Player X's purpose is to minimize the functional

$$J = F(x(T), y(T)) \tag{1.4}$$

where $F(x, y)$ is a specified function. Player Y realizes his own control $v(t)$ and the signal $q(t)$ as time functions subject to constraints (1.1) – (1.3) and counteracts the intentions of player X. On the quantities $u(z)$, $v(t)$, $q(t)$ and relations (1.1) we impose constraints ensuring the existence and uniqueness of absolutely continuous solutions of system (1.1); if necessary, we also include the vector $v(q(t))$ into collection z. Considering the game from the positions of player X, we pose the following problem.

Problem 1. Find the noise distribution P^* worst for player X and his optimal minimax strategy u^*, i.e. the strategy and the set P^* satisfying the relation

$$J^* = \min_u \max_P \sup_v J[u, v, P] = \sup_v J[u^*, v, P^*] \tag{1.5}$$

Find the minimal guaranteed value J^* of functional (1.4).

Here $J[u, v, P]$ is a value of functional (1.4), corresponding to strategy u, control v and noise set P. The extrema in (1.5) are taken over the sets described above. By realizing the various sets P of noise instants, player Y cannot increase the maximum of functional (1.4) guaranteed for himself, but can, in general, increase the minimum of this functional, guaranteed for player X as compared with a noise-free game ($P = \phi$).

Let us point out a case when Problem 1 is reduced to a problem with a fixed noise set. We present its formulation. Let set P (the signal $q(t)$, $t \in [t^\circ, T]$) be specified before the start of the game and be known to player X who employs strategies in the form of functions of $z' = \{x(t), y(q(t)), t\}$.

Problem 2. Find the optimal minimax strategy u° of player X

$$J^\circ = \min_u \sup_v J[u, v] = \sup_v J[u^\circ, v] \tag{1.6}$$

Find the quantity $J^\circ = J^\circ[P]$.

Suppose that a solution of Problem 2 exists for all admissible sets P and has the form $u^\circ[t] = u^\circ(z', P) = u^\circ(z', q(t))$, i.e. for constructing an optimal control at instant t by player X, it is sufficient that he has only local information on set P consisting of knowning the signal $q(t)$. Then the noise distribution P^*, worst for player X in Problem 1, can be found from the condition

$$J^* = \max_P J^\circ[P] = J^\circ[P^*] \tag{1.7}$$

and the strategy $u^*(z) = u^\circ(z', q(t))$ is the solution of Problem 1. The maximum in (1.7) is taken over admissible sets P.

Let us prove the assertions made. Suppose that under the hypotheses of Problem 1 player X employs a strategy $u = u^\circ(z', q(t))$. The equality

$$\sup_v J[u^\circ, v, P] = J^\circ[P]$$

is valid for arbitrary P realized by player Y, whence it follows:

$$J^* \leqslant \max_P \sup_v J[u^\circ, v, P] = \max_P J^\circ[P] \tag{1.8}$$

On the other hand, for each fixed P we have

$$J^* \geqslant J^\circ[P] \tag{1.9}$$

since player Y has the possibility in Problem 1 of realizing precisely this set P, while the quantity $J^\circ[P]$ is the minimal guaranteed value of functional (1.4) for the specified noise set P. Carrying out a maximization over P in (1.9) and allowing for (1.8), we obtain equality (1.7).

2. Examples. Let us consider in parallel four encounter games with noise, whose solutions can be described in common. Let the players X and Y be moved on the interval $[0, T]$ in accorcance with one of the pairs of equations of motion and constraints

X:	Y:	
$x^\cdot = u, \ \lvert u \rvert \leqslant \mu;$	$y^\cdot = v, \ \lvert v \rvert \leqslant \nu$	(2.1)
$x^{\cdot\cdot} = u, \ \lvert u \rvert \leqslant \mu;$	$y^{\cdot\cdot} = v, \ \lvert v \rvert \leqslant \nu$	(2.2)
$x^{\cdot\cdot} = u, \ \lvert u \rvert \leqslant 1;$	$y^\cdot = v, \ \lvert v \rvert \leqslant 1$	(2.3)

$$x^{\cdot} = \alpha x + u, \quad |u| \leqslant 1; \qquad y^{\cdot} = \lambda y + v, \quad |v| \leqslant \nu \tag{2.4}$$
$$\mu > \nu > 0, \quad \alpha = \pm 1, \quad -\infty < \lambda < +\infty$$

Here x, y, u, v are vectors of arbitrary identical dimension; μ, ν, α, λ are real parameters. The initial values of the phase vectors and velocities (for the equations with second time derivative) are specified by the equalities

$$x(0) = x^{\circ}, \quad x^{\cdot}(0) = x_1^{\circ}, \quad y(0) = y^{\circ}, \quad y^{\cdot}(0) = y_1^{\circ} \tag{2.5}$$

The conditions of information available to player X and the admissible noise sets P are described in Sect. 1. For the games (2.1) – (2.4) we pose Problem 1 with the functional

$$J = |x(T) - y(T)| \tag{2.6}$$

Let us show that the examples (2.1) – (2.4) being examined refer to the case mentioned at the end of Sect. 1, i.e. can be solved by reduction to Problem 2 and by the use of relation (1.7).

Solution of Problem 2. The quantity (1.6) can be constructed for (2.1)–(2.6), having solved the corresponding multistage-differential equations of Bellman for multistage-differential games, to which Problem 2 is reduced (see [1, 4]). A technically simpler way, to which we give preference below, consists (with some loss of strictness) in reducing the examples being analyzed to a multistage game with a finite or denumerable number of steps.

In what follows we shall need sufficient sets of observation instants $A_T \subset [0, T]$ (see [4]). The observations of player X at the points of this discrete set ensure him the same guaranteed minimum of functional (2.6) as in a game with complete information. Sets $A_T = \{t_i\}$ for games (2.1), (2.2) are constructed in [1] and consist, respectively, of the denumerable set of points

$$t_i = T[1 - (\nu / \mu)^{i-1}], \quad t_i = T[1 - (\nu / \mu)^{(i-1)/2}], \quad i = 1, 2, \ldots \tag{2.7}$$

condensing to the instant $t^* = T$. In game (2.3) the points of set A_T are given by the recurrence relations ([4])

$$T - t_i - (T - t_{i+1})^2 / 2 = h, \quad i = 0, \pm 1, \pm 2, \ldots, \tag{2.8}$$
$$t_1 = 0, \quad t_0 = T$$
$$h = 1/2, \quad T > 1; \quad h = T(1 - T/2), \quad 0 < T \leqslant 1$$

The condensation points of (2.8) are the instant $t^* = T - 1$ for $T > 1$ and the instant $t^* = 0$ for $0 < T \leqslant 1$. The set A_T for game (2.4) is constructed in [5]; depending on the values of the parameters the single condensation point t^* of this set can be located at the origin, at the end, or inside the interval of motion.

Let us consider the discrete set of observation instant $A_0 = (A_T \cap Q) \cup \bar{P} \setminus P$ ($\bar{P}$ is the closure of set P), $A_0 \subset Q$. Set A_0 is obtained by supplementing the point of set A_T by those boundary points of region P which fall into region Q and is finite (denumerable or finite) if $t^* \notin Q$ ($t^* \in Q$). The points of set A_0 are denoted τ_k, $k = 0, \pm 1, \pm 2, \ldots$, numbering them by positive indices in order of succession from left to right, $\tau_1 = 0$, and by negative indices, from right to left, $\tau_0 = T$. Depending upon the location of point t^* the sets of positive and negative indices can be finite, while points with negative indices may be absent altogether. By K we denote the set of indices corresponding to the points of A_0, excepting the zero index corresponding to the instant $\tau_0 = T$, i.e. $A_0 = \{\tau_k: k \in K + 0\}$.

Using the substitutions

$$x(t) + (T - t)\, x^\cdot(t) \to x(t), \quad x^\circ + T x_1^\circ \to x^\circ$$

$$y(t) + (T - t)\, y^\cdot(t) \to y(t), \quad y^\circ + T y_1^\circ \to y^\circ$$

we write the equations of motion of players X in (2.2), (2.3) and Y in (2.2) with initial phase vector values as

$$x^\cdot = (T - t)\, u, \quad x(0) = x^\circ, \quad y^\cdot = (T - t)\, v, \quad y(0) = y^\circ \qquad (2.9)$$

The form of functional (2.6) does not alter here since the values of the new phase vectors at $t = T$ are identical with the values of the original ones. For the three equations indicated, by $x(t)$, $y(t)$ we shall understand the linear combination of phase vector and velocity introduced above.

Suppose that in games (2.1) – (2.4), (2.9) the player X observes the position $\{x_k, y_k\}$, $x_k = x(\tau_k)$, $y_k = y(\tau_k)$, $k \in K + 0$, only at points of set A_0 and specifies his own control on the intervals $[\tau_k, \tau_{k+1})$, $k \in K$, in the form of the functions $u = u(x_k, y_k; t)$ integrable in time and satisfying constraints (2.1) – (2.4), i.e. employs piecewise-program strategies. By an integration of Eqs. (2.1) – (2.4), (2.9) we can obtain

$$x_{k+1} = r_k x_k + p_k u_k, \quad |u_k| \leqslant 1 \qquad (2.10)$$
$$y_{k+1} = s_k y_k + q_k v_k, \quad |v_k| \leqslant 1, \quad k \in K$$

where the vectors u_k, v_k are constructed according to the strategy and control specified (see [4]). The coefficients of Eqs. (2.10) for games (2.1) – (2.4), respectively, have the form

$$r_k = s_k = 1, \quad p_k = \mu\Delta_k, \quad q_k = \nu\Delta_k$$
$$r_k = s_k = 1, \quad p_k = \mu\alpha_k, \quad q_k = \nu\alpha_k$$
$$r_k = s_k = 1, \quad p_k = \alpha_k, \quad q_k = \Delta_k$$
$$r_k = e^{\alpha\Delta_k}, \quad s_k = e^{\lambda\Delta_k}, \quad p_k = (e^{\alpha\Delta_k} - 1) / \alpha$$
$$q_k = \nu\,(e^{\lambda\Delta_k} - 1) / \lambda$$
$$\alpha_k = \Delta_k (T - \tau_k - \Delta_k / 2), \quad \Delta_k = \tau_{k+1} - \tau_k, \quad k \in K$$

In the notation adopted the initial values and the functional (2.6) for (2.10) are written as

$$x_1 = x^\circ, \quad y_1 = y^\circ; \quad J = |x_0 - y_0| \qquad (2.11)$$

Relations (2.10), (2.11) specify a multistage game. In this game the collection of functions $u_k = u_k(x_k, y_k)$, $|u_k| \leqslant 1$, $k \in K$ is called an admissible strategy U_Δ of player X . In (2.10), (2.11) player Y realizes the control V_Δ in the form of the sequence v_k, $|v_k| \leqslant 1$, $k \in K$. We note that certain constraints, following from the continuity of the phase trajectories in the original games, have been imposed on the solution of (2.10) (trajectories.

Let us clarify what we have said for the case when K contains a countable number of positive and negative indices. In this case a solution of (2.10) is the collection

$\{x_k\}, \{y_k\}$, $k \in K$, satisfying (2.10) and the conditions $x_k, x_{-k} \to x^*$; $y_k, y_{-k} \to y^*$ as $k \to \infty$, where x^*, y^* correspond to $x(t^*)$, $y(t^*)$ in the original equations. By the construction of (2.10) we see that if in the original differential games with observations on A_0 we adopt piecewise-constant strategy $u(x_k, y_k; t) \equiv u_k(x_k, y_k)$ and control $v(t) \equiv v_k$, $t \in [\tau_k, \tau_{k+1})$, $k \in K$, corresponding to some U_Δ, V_Δ from (2.10), then the sequence $x(\tau_k)$, $y(\tau_k)$ realized is just the same as in (2.10) with the pair U_Δ, V_Δ.

Problem 2*. Find the minimax optimal strategy U_Δ^* of player X in game (2.10), (2.11), i.e. the strategy satisfying the relation

$$J_0 = \min_{U_\Delta} \max_{V_\Delta} J[U_\Delta, V_\Delta] = \max_{V_\Delta} J[U_\Delta^*, V_\Delta] \tag{2.12}$$

Find the quantity of J_0.

We define the Bellman function

$$S_k(x_k, y_k) = \min_{U_\Delta} \max_{V_\Delta} J, k \in K, \quad S_0(x_0, y_0) = |x_0 - y_0| \tag{2.13}$$

Here, in the minimum (maximum) operations there participate those components u_i (v_i) of strategy U_Δ (of control V_Δ), for which the points τ_i lie to the right of τ_k, i.e. $1/i \leqslant 1/k$, $i \in K$; in addition, we assume that the position $\{x_k, y_k\}$ is realized at instant τ_k. From (2.12), (2.13) we have $J_0 = S_1(x^\circ, y^\circ)$. The recurrence relation for the Bellman function also follows from (2.13):

$$S_k(x_k, y_k) = \min_{u_k} \max_{v_k} S_{k+1}(x_{k+1}, y_{k+1}) \tag{2.14}$$
$$k \in K, \quad S_0(x_0, y_0) = |x_0 - y_0|$$

where x_{k+1}, y_{k+1} are taken in the form (2.10). It is easy to verify that the unique solution of relations (2.14) is

$$S_k(x_k, y_k) = \max[\Phi_k, R_k(x_k, y_k)] \tag{2.15}$$
$$\Phi_k = \max R(\tau_i, \tau_{i+1}), \quad k \in K \qquad 1/i \leqslant 1/k, \quad i \in K$$

Here the function $R(\xi, \eta)$ for games (2.1) – (2.4), respectively, equals

$$\nu(T - \xi) - \mu(T - \eta), \quad [\nu(T - \xi)^2 - \mu(T - \eta)^2]/2 \tag{2.16}$$
$$T - \xi - (T - \eta)^2/2, \qquad \nu(e^{\lambda(T-\xi)} - 1)/\lambda - (e^{\alpha(T-\eta)} - 1)/\alpha$$

Further, $R_k(x_k, y_k) = |w_k(x_k, y_k)| + R(\tau_k, \tau_k)$, where for the first three games $w_k = x_k - y_k$, while for game (2.4)

$$w_k = e^{\alpha(T-\tau_k)} x_k - e^{\lambda(T-\tau_k)} y_k$$

The components of strategy U_Δ^* of Problem 2*, imparting the minimum in (2.14), have the form

$$u_k = -w_k / |w_k|, \qquad |w_k| > p_k l_{k+1} \tag{2.17}$$
$$u_k = -w_k / (p_k l_{k+1}), \quad |w_k| \leqslant p_k l_{k+1}, \quad k \in K$$

The quantities $l_k = 1$ in the first three games and $l_k = e^{\alpha(T-\tau_k)}$ in game (2.4); the p_k are taken from (2.10). Thus, if in (2.1) – (2.4), (2.9) we substitute the piecewise-constant strategy (2.17), then the value

$$J_0 = S_1(x^\circ, y^\circ) = \max[\Phi_1, R_1(x^\circ, y^\circ)] \tag{2.18}$$

of functional (2.6) is guaranteed. The inconvenience of strategy (2.17) is that knowledge of the instant τ_{k+1} at the instant τ_k is assumed in it. Therefore, let us describe another strategy equivalent to (2.17) and relaying only on the last observation instant $q(t)$ (cf (1.3))

$$q(t) = \tau_k, \quad t \in [\tau_k, \tau_{k+1}), \quad k \in K \qquad (1.19)$$

Let $w(t) = x(t) - y(q(t))$ for (2.1) – (2.3), (2.9) and

$$w(t) = e^{\alpha(T-t)}x(t) - e^{\lambda(T-q(t))}y(q(t))$$

for (2.4). Then it is easy to verify that the piecewise-program strategy

$$u = -w(t) / |w(t)|, \quad w(t) \neq 0; \quad u = 0, \quad w(t) = 0 \qquad (2.20)$$

realizes in (2.1) – (2.4), (2.9) the same sequence x_k, y_k, that (2.17) does, and consequently, guarantees the quantity (2.18) for (2.6). Generally speaking, strategy (2.20) differs from the strategies introduced in Sect. 1, but less information is required for constructing (2.20) (continuous observations on the intervals $[a_i, b_i]$ are replaced by discrete ones). Therefore, we assume that strategies of form (2.20) are admissible in Problem 2. Furthermore, strategy (2.20) satisfies the last assumption in Sect. 1.

It can now be shown (for the application of (1.7)) that $J_0 = J^\circ$, i.e. observations on $A_0 \subset Q$ in the examples analyzed guarantee the same value of functional (2.6) as do the observations on Q. To show this we note that any pair $\tau_k, \tau_{k+1} \in A_0$ is one of the four pairs

$$b_i, \ a_{i+1}; \quad t_i, \ t_{i+1}; \quad t_i, \ b_j \ (t_{i+1} > b_j); \quad a_i, \ t_{j+1} \ (a_i > t_j) \qquad (2.21)$$

for some values of i, j. Here a_i, b_i are taken from (1.2), $t_i \in A_T$. It can be verified ((2.7), (2.8), [5]) that the points of A_T satisfy the condition

$$R(t_i, t) < R_*, \quad R(t, t_{i+1}) < R_*, \quad t_i < t < t_{i+1} \qquad (2.22)$$

$$R(t_i, t_{i+1}) = R_* = \max R(t, t), \ t \in [0, T]$$

where R has form (2.14). From (2.21), (2.22), (2.15) we have

$$S_1(x^\circ, y^\circ) = \max [R_*, \max_{1 \leqslant i \leqslant N-1} R(b_i, b_i + \vartheta_i), R_1(x^\circ, y^\circ)] \qquad (2.23)$$

From (2.22), (2.15) we see that supplementing A_0 with an arbitrary finite number of discrete observation instants from Q does not improve the guaranteed result (2.23). This argument enables us to write $J_0 = J^\circ$.

Construction of the optimal noise set. To seek the optimal set P^* we accept that in (1.7) the quantity J° is of form (2.23). When computing the maximum over admissible sets P (over parameters b_i, ϑ_i of form (1.2) only the second alternative in (2.23) is affected, i.e. P^* should be sought from the condition

$$\max_{\{b_i, \vartheta_i\}} \max_{1 \leqslant i \leqslant N-1} R(b_i, b_i + \vartheta_i) \qquad (2.24)$$

Further, we note, that as follows from (1.2), for any admissible set P the pair of numbers $\xi = b_i$, $\eta = b_i + \vartheta_i$, $i = 1, \ldots, N-1$ belongs to the set

$$G = \{(\xi, \eta): \xi \geqslant 0; \ \eta \leqslant T; \ 0 \leqslant \eta - \xi \leqslant \vartheta\} \qquad (2.25)$$

The first two constraints in (2.25) signify that the starting and ending instants of the noise interval lie on the interval $[0, T]$; the third constraint signifies that duration of each particular noise is nonnegative and does not exceed the total duration ϑ. Consequently, the maximum in (2.24) does not exceed the quantity

$$R^* = \max R(\xi, \eta), \quad (\xi, \eta) \in G \tag{2.26}$$

The constant R_* is found by maximizing function R for $\xi - \eta = 0$, $0 \leqslant \xi \leqslant T$, i.e. on a subset of set G; therefore, $R_* \leqslant R^*$. We see that if we take all the ϑ_i equal to zero except for some one value (say, ϑ_1), then each point of region G in (2.25) corresponds to some admissible realization of noise set P. This assumption indicates that player Y realizes only one noise interval (b_1, a_2). Suppose that the maximum (2.26) is reached at the point (ξ^*, η^*). Then from what has been presented it follows that the optimal noise set consists of one open interval $P^* = (b_1^*, a_2^*)$, $b_1^* = \xi^*$, $a_2^* = \eta^*$, while the quantity (1.5) equals

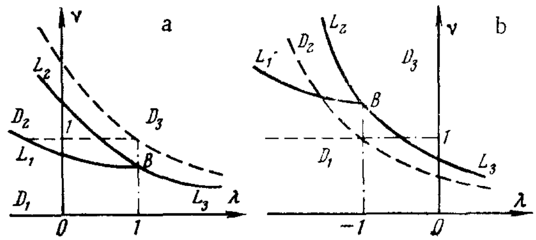

Fig. 1

$$J^* = \max [R^*, R_1] \tag{2.27}$$

Thus, the problem of determining the optimal noise set P^* and the minimal guaranteed value J^* of the functional is reduced, in the examples being analyzed, to the determination of the maximum point of functions (2.16) over region (2.25) and to the computation of the maximal value of these functions. We note that in all four examples the desired maximum is achieved on the segment $\eta - \xi = \vartheta$, $0 \leqslant \xi \leqslant T - \vartheta$, being a boundary segment for region G, i.e. $\eta^* - \xi^* = a_2^* - b_1^* = \vartheta$. This circumstance is sufficiently obvious and signifies that it is advantageous for player Y to expend the whole noise resources.

We describe the optimal noise sets, omitting the computations. For game (2.1) it has the form

$$P^* = (T - \vartheta, T), \quad b_1^* = T - \vartheta, \quad a_2^* = T \tag{2.28}$$

i.e. it is advantageous to player Y to switch on the noise on the last segment of motion. In games (2.2), (2.3) the optimal noise interval is located (depending on the quantity ϑ) on the middle or the initial segment of the motion interval. The endpoints of the interval are determined by the relations (2.29)

$$b_1^* = T - \vartheta / (1 - \nu / \mu), \quad a_2^* = b_1^* + \vartheta, \quad 0 \leqslant \vartheta \leqslant (1 - \nu / \mu) T$$
$$b_1^* = 0, \quad a_2^* = \vartheta, \quad (1 - \nu / \mu) T < \vartheta \leqslant T$$

for game (2.2) and by the relations

$$b_1^* = T - \vartheta - 1, \quad a_2^* = T - 1, \quad 0 \leqslant \vartheta \leqslant T - 1 \tag{2.30}$$
$$b_1^* = 0, \quad a_2^* = \vartheta, \quad T - 1 \leqslant \vartheta \leqslant T$$

for game (2.3).

Depending on the problem parameters, all three cases of location of the noise interval can be realized in game (2.4). The Figure shows the plane of parameters λ, ν for $\alpha = 1$ (Fig. 1 a) and for $\alpha = -1$ (Fig. 1 b). The curves L_i, $i = 1,2,3$

$$L_1: \quad \nu = e^{-\vartheta\lambda}, \qquad \lambda < \alpha, \quad L_2: \nu = e^{\alpha(T-\vartheta)-\lambda T}, \quad \lambda < \alpha$$
$$L_3: \nu = \lambda\, (e^{\alpha(T-\vartheta)} - 1) / \alpha\, (e^{\lambda T} - e^{\lambda\vartheta}), \quad \lambda > \alpha$$

partition this plane into three open regions D_i, $i = 1,2,3$. The dotted lines show the curves L_i for $\vartheta = 0$ (see the figure in [5]). In region D_1 the optimal noise interval is located at the right end of the interval $[0, T]$: $b_1^* = T - \vartheta$, $a_2^* = T$; in region D_3 at the left end: $b_1^* = 0$, $a_2^* = \vartheta$. In region D_2 the set P^* is located strictly inside the interval $[0, T]$ and has as its endpoints

$$b_1^* = T - \frac{\ln \nu + \alpha\vartheta}{\alpha - \lambda}, \qquad a_2^* = T - \frac{\ln \nu + \lambda\vartheta}{\alpha - \lambda} \tag{2.31}$$

If the pair $(\lambda, \nu) \in L_3$, the set P^* is determined ambiguously and is located at the right or the left end of the motion interval. Finally, at point B, $\lambda = \alpha$, $\nu = e^{-\alpha\vartheta}$, and set P^* can be located arbitrarily, i.e. $0 \leqslant b_1^* \leqslant T - \vartheta$, $a_2^* = b_1^* + \vartheta$. The quantity R^* in (2.27) is found by substituting the points (2.28)–(2.31) into function (2.16). As an example we present the value of R^* in games (2.1), (2.3)

$$R^* = \nu\vartheta, \qquad R^* = \begin{cases} \vartheta + {}^1/_2, & 0 \leqslant \vartheta \leqslant T - 1 \\ T - {}^1/_2 (T - \vartheta)^2, & T - 1 \leqslant \vartheta \leqslant T \end{cases}$$

Note 1. Let us denote by $P^*[\vartheta]$ the optimal noise set corresponding to resource ϑ. It can be verified that sets (2.28), (2.29) are such that $P^*[\vartheta] \subset P^*[\vartheta']$ for $\vartheta \leqslant \vartheta'$. On the other hand, in (2.29) we can find $\vartheta \leqslant \vartheta'$ (ϑ is sufficiently small) such that $P^*[\vartheta] \cap P^*[\vartheta'] = \phi$. Thus, as the resource ϑ increases the optimal noise interval P^* does not simply extend but also shifts along the interval $[0, T]$.

Note 2. A comparison of (2.28)–(2.31) with results in [1, 4, 5] shows that it is not always advantageous to player Y to switch the noise on in a neighborhood of the condensation point of the sufficient observation instants set A_T. This signifies that observations in a neighborhood of the condensation point are important for an exact achievement of the minimal magnitude of the functional; the magnitude itself of the functional is influenced, generally speaking, by observations at other points which are excluded due to the optimal noise set.

Note 3. It was shown in [4, 5] that to ensure the guaranteed maximum of functional (1.4) it is necessary for player Y to have the observation at an interior point of the interval $[0, T]$. Consequently, some noise interval, containing this point, can decrease the guaranteed maximum of the functional, i.e. a meaningful maximin reformulation of the statement in Sect. 1 is possible.

REFERENCES

1. Melikian, A. A. and Chernous'ko, F. L., On differential games with variable informativeness conditions. Dokl. Akad. Nauk SSSR, Vol. 203, № 1, 1972.
2. Krasovskii, N. N., Game Problems on the Contact of Motions. Moscow, "Nauka", 1970.
3. Patsko, V. S., A model example of a pursuit game problem with incomplete information. Differentsial'nye Uravneniia, Vol. 7, № 3, 1971.
4. Melikian, A. A., On minimal observations in a game of encounter. PMM Vol. 37, № 3, 1973.
5. Melikian, A. A., On the minimal number of observation instants in a model game of encounter. Izv. Akad. Nauk ArmSSR, Matematika, Vol. 9, № 3, 1974.

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